数理解析研究所講究録
1112巻 1999年 113-123



# UNIVALENCE AND STARLIKENESS OF SOLUTIONS OF $W'' + aW' + bW = 0$

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We consider the differential equation

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0,$$

where $a(z)$ and $b(z)$ are analytic in the unit disc $\Delta$. In this paper, we show that the above differential equation has a solution $w(z)$ univalent and starlike in $\Delta$ under some conditions. It is related to results of S. S. Miller and M. S. Robertson.

## 1. Introduction

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be an analytic function defined in the unit disc $\Delta = \{z : |z| < 1\}$. We denote the class of such functions by $A$. If in addition $f(z)$ is univalent, then we say $f(z) \in S$. Suppose $f'(z) \neq 0$ in $\Delta$, then we define

$$S(f, z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}(z)\right)^2$$

to be the *Schwarzian derivative* of $f(z)$.

Our starting point is the following result of S. S. Miller.

**Theorem A** (Miller [4]). *Let $p(z)$ be analytic in the unit disc $\Delta$ with $|zp(z)| < 1$. Let $v(z)$, $z \in \Delta$, be the unique solution of*

$$v''(z) + p(z)v(z) = 0 \tag{1.1}$$

*with $v(0) = 0$ and $v'(0) = 1$. Then*

$$\left|\frac{zv'(z)}{v(z)} - 1\right| < 1, \tag{1.2}$$

*and $v(z)$ is a starlike conformal map of the unit disc.*

Theorem A is related to the next results of M. S. Robertson and Z. Nehari.

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**Theorem B** (Robertson [8]). *Let $zp(z)$ be analytic in $\Delta$ and*

$$\text{(1.3)} \qquad \operatorname{Re}\{z^2 p(z)\} \leq \frac{\pi^2}{4}|z|^2 \qquad (z \in \Delta).$$

*Then the unique solution $W = W(z)$, $W(0) = 0$, $W'(0) = 1$ of*

$$\text{(1.4)} \qquad W''(z) + p(z)W(z) = 0$$

*is univalent and starlike in $\Delta$. The constant $\pi^2/4$ is best possible one.*

**Theorem C** (Nehari [6]). *If $f(z) \in A$ and it satisfies*

$$\text{(1.5)} \qquad |S(f,z)| \leq \frac{\pi^2}{2} \qquad (z \in \Delta),$$

*then $f(z)$ is univalent. The result is sharp.*

**Remark 1.** The constant $\pi^2/2$ is best possible as shown by the example $\frac{e^{i\pi z}-1}{i\pi}$. We note that by putting $p(z) = \frac{1}{2}S(f,z)$ in (1.3) of Theorem B, then (1.5) of Theorem C implies (1.3). Therefore, Nehari's theorem has a stronger hypothesis. Thus Robertson proved that the unique solution of the equation (1.4) is starlike whereas Nehari proved the quotient of the linearly independent solution of (1.4) is univalent.

We also have

**Theorem D** (Gobriel [2]). *Suppose $f(z) \in A$ and that*

$$\text{(1.6)} \qquad |S(f,z)| \leq 2c_0 \approx 2.73 \qquad (z \in \Delta),$$

*where $c_0$ is the smallest positive root of the equation $2\sqrt{x} - \tan\sqrt{x} = 0$, then $f(z)$ maps $\Delta$ onto a starlike domain.*

Recall that $f(z) \in S$ is starlike with respect to the origin if and only if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ for all $z \in \Delta$. We denote the class of starlike functions by $S^*$.

## 2. A class of bounded functions

Let $B_J$ denote the class of bounded functions $w(z) = w_1 z + w_2 z^2 + \cdots$ analytic in the unit disc $\Delta$ for which $|w(z)| < J$. If $g(z) \in B_J$, then by using the Schwarz lemma we can show that the function $w(z)$ defined by $w(z) = z^{-\frac{1}{2}} \int_0^z g(t) t^{-\frac{1}{2}} dt$ is also in $B_J$. Writing this result in terms of derivatives we have

$$\text{(2.1)} \qquad \left|\frac{1}{2}w(z) + zw'(z)\right| < J \quad (z \in \Delta) \quad \Rightarrow \quad |w(z)| < J \quad (z \in \Delta).$$

If we let $h(u,v) = \frac{1}{2}u + v$ we can write (2.1) as

$$|h(w(z), zw'(z))| < J \quad \Rightarrow \quad |w(z)| < J. \tag{2.2}$$

In this section, we will show that (2.2) holds for functions $h(u,v)$ satisfying the following definition.

**Definion 1.** Let $H_J$ be the set of complex functions $h(u,v)$ satisfying:
(i) $h(u,v)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C}$,
(ii) $(0,0) \in D$ and $|h(0,0)| < J$,
(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J$ when $(Je^{i\theta}, Ke^{i\theta}) \in D$, $\theta$ is real and $K \geq J$.

**Example 1.** It is easy to check that the following function $h(u,v)$ is in $H_J$:
$h(u,v) = \alpha u + v$ where $\alpha$ is complex with $\mathrm{Re}\alpha \geq 0$, and $D = \mathbb{C} \times \mathbb{C}$.

**Definition 2.** Let $h \in H_J$ with corresponding domain $D$. We denote by $B_J(h)$ those functions $w(z) = w_1 z + w_2 z^2 + \cdots$ which are analytic in $\Delta$ satisfying
(i) $(w(z), zw'(z)) \in D$,
(ii) $|h(w(z), zw'(z))| < J \qquad (z \in \Delta)$.

The set $B_J(h)$ is not empty since for any $h \in H_J$ it is true that $w(z) = w_1 z \in B_J(h)$ for $|w_1|$ sufficiently small depending on $h$.

We need the following lemma to prove our results.

**Lemma 1** (Miller and Mocanu [5]). *Let $w(z) = w_1 z + w_2 z^2 + \cdots$ be analytic in $\Delta$ with $w(z) \not\equiv 0$. If $z_0 = r_0 e^{i\theta_0}$, $0 < r_0 < 1$, and $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$, then*
(i)

$$\frac{z_0 w'(z_0)}{w(z_0)} = m$$

*and*
(ii)

$$\mathrm{Re}\left[\frac{z_0 w''(z_0)}{w'(z_0)}\right] + 1 \geq m$$

*where $m \geq 1$.*

**Theorem 1.** *For any $h \in H_J$, $B_J(h) \subset B_J$.*

*Proof.* Let $w(z) \in B_J(h)$. Suppose that $\exists z_0 = r_0 e^{i\varphi_0} \in \Delta$ $(0 < r_0 < 1)$ such that

$$\max_{|z| \leq r_0} |w(z)| = |w(z_0)| = J.$$

Then $w(z_0) = Je^{i\theta}$ and since by Lemma

$$\frac{z_0 w'(z_0)}{w(z_0)} = m \geq 1,$$

we have

$$z_0 w'(z_0) = Ke^{i\theta} \qquad (K = mJ \geq J)$$

and thus

$$h(w(z_0), z_0 w'(z_0)) = h(Je^{i\theta}, Ke^{i\theta}).$$

Since $h \in H_J$ this implies that

$$|h\,(w(z_0), z_0 w'(z_0))\,| \geq J$$

which contradiction of $w(z) \in B_J(h)$. Hence $|w(z)| < J$ $(z \in \Delta)$, and thus $w(z) \in B_J$.

**Remark 2.** In other words, above theorem shows that if $h \in H_J$, with corresponding domain $D$ and if $w(z) = w_1 z + w_2 z^2 + \cdots$ is analytic in $\Delta$ and $(w(z), zw'(z)) \in D$, then

$$|h(w(z), zw'(z))| < J \quad \Rightarrow \quad |w(z)| < J.$$

Furthermore, Theorem 1 can be used to show that certain first order differential equations have bounded solutions. The proof of the following theorem follows immediately from Theorem 1.

**Theorem 2.** *Let $h \in H_J$ and $b(z)$ be a analytic function in $\Delta$ with $|b(z)| < J$.If the differential equation*

$$h(w(z), zw'(z)) = b(z) \qquad (w(0) = 0)$$

*has a solution $w(z)$ analytic in $\Delta$, then $|w(z)| < J$.*

## 3. Main results

Our main result is the following theorem.

**Theorem 3.** *Let $a(z)$ and $b(z)$ be analytic in $\Delta$ with $\left|z\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}a^2(z)\right)\right| < \frac{1}{2}$ and $|a(z)| < 1$. Let $w(z)$ $(z \in \Delta)$ be the solution of the following second order linear differential equation*

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0 \tag{3.1}$$

*with $w(0) = 0$, $w'(0) = 1$. Then $w(z)$ is starlike in $\Delta$.*

*Proof.* The transformation

$$w(z) = \exp\left(-\frac{1}{2}\int_0^z a(\xi)d\xi\right) v(z) \tag{3.2}$$

leads to the normal form

$$v''(z)+\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}a^2(z)\right)v(z)=0 \tag{3.3}$$

and $v(0)=0$, $v'(0)=1$. If we put

$$u(z)=\frac{zv'(z)}{v(z)}-1 \qquad (z\in\Delta), \tag{3.4}$$

then $u(z)$ is analytic in $\Delta$, $u(0)=0$ and (3.3) becomes

$$u^2(z)+u(z)+zu'(z)=-z^2\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}a^2(z)\right), \tag{3.5}$$

or equivalently

$$h(u(z),zu'(z))=-z^2\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}a^2(z)\right), \tag{3.6}$$

where $h(u,v)=u^2+u+v$. It is easy to check $h(u,v)\in H_{\frac{1}{2}}$. i.e.,

(i) $h(u,v)$ is continuous in $D=\mathbb{C}\times\mathbb{C}$,
(ii) $(0,0)\in D$, $|h(0,0)|=0<\frac{1}{2}$,
(iii) $\left|h\left(\frac{1}{2}e^{i\theta},Ke^{i\theta}\right)\right|\geq\frac{1}{2}$ $\qquad\left(K\geq\frac{1}{2}\right)$.

From assumption, we have

$$\left|-z^2\left(b(z)-\frac{1}{2}a'(z)-\frac{1}{4}a^2(z)\right)\right|<\frac{1}{2} \qquad (z\in\Delta).$$

By using Theorem 2, we have

$$|u(z)|<\frac{1}{2} \qquad (z\in\Delta).$$

Therefore, we obtain

$$\left|\frac{zv'(z)}{v(z)}-1\right|<\frac{1}{2} \qquad (z\in\Delta).$$

This implies that

$$\frac{1}{2}<\mathrm{Re}\left\{\frac{zv'(z)}{v(z)}\right\}<\frac{3}{2} \qquad (z\in\Delta). \tag{3.7}$$

From (3.2), we have

$$\exp\left(\frac{1}{2}\int_0^z a(\xi)d\xi\right)w(z)=v(z). \tag{3.8}$$

Logarithmically differentiating of (3.8) leads to

$$\frac{zw'(z)}{w(z)}=\frac{zv'(z)}{v(z)}-\frac{z}{2}a(z). \tag{3.9}$$

Combining (3.9) and $|a(z)|<1$, we obtain

$$\mathrm{Re}\left\{\frac{zw'(z)}{w(z)}\right\}\geq\mathrm{Re}\left\{\frac{zv'(z)}{v(z)}\right\}-\frac{1}{2}|za(z)|>0 \qquad (z\in\Delta),$$

and thus $w(z)$ is starlike in $\Delta$.

**Example 2.** Let $a(z) = -z$, $b(z) = \frac{z^2}{4}$ in Theorem 3, then the solution of

$$w''(z) - zw'(z) + \frac{z^2}{4}w(z) = 0 \tag{3.10}$$

is

$$w(z) = \sqrt{2}e^{\frac{z^2}{4}} \sin\frac{z}{\sqrt{2}} \in S^*.$$

Let $a(z) = -z$, $b(z) = \lambda$ ($\lambda \in \mathbb{C}$) in Theorem 3, then differential equation (3.1) is

$$w''(z) - zw'(z) + \lambda w(z) = 0. \tag{3.11}$$

The differential equation (3.11) is called Hermite's differential equation.

By the transformation $w(z) = e^{\frac{z^2}{4}}v(z)$, (3.11) lead to

$$v''(z) + \left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right)v(z) = 0. \tag{3.12}$$

This differential equation is well-known, that is, *Weber's equation* (see [9]).

**Theorem 4.** *We consider Weber's differential equation* (3.12). *Let* $\left|\lambda + \frac{1}{2} - \frac{z^2}{4}\right| < 1$, *then the solution* $v(z)$ *is starlike in* $\Delta$.

*Proof.* We put

$$u(z) = \frac{zv'(z)}{v(z)} - 1. \tag{3.13}$$

Then $u(z)$ is analytic in $\Delta$, $u(0) = 0$ and

$$u^2(z) + u(z) + zu'(z) = -z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) \tag{3.14}$$

or equivalently

$$h(u(z), zu'(z)) = -z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right), \tag{3.15}$$

where $h(u, v) = u^2 + u + v$. It is easy to check $h(u, v) \in H_1$, i.e.,

(i) $h(u, v)$ is continuous in $D = \mathbb{C} \times \mathbb{C}$,
(ii) $(0, 0) \in D$, $|h(0, 0)| = 0 < 1$,
(iii) $|h(e^{i\theta}, Ke^{i\theta})| \geq 1 \qquad (K \geq 1)$.

From assumption we have

$$\left|-z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right)\right| < 1 \qquad (z \in \Delta).$$

By using Theorem 2, we obtain

$$|u(z)| < 1 \qquad (z \in \Delta).$$

Therefore, this shows that

$$\left|\frac{zv'(z)}{v(z)} - 1\right| < 1,$$

which implies $\mathrm{Re}\left\{\frac{zv'(z)}{v(z)}\right\} > 0$ $(z \in \Delta)$, that is, $v(z)$ is starlike in $\Delta$.

**Remark 3.** The solutions of Weber's differential equation $v''(z)+\left(\lambda+\frac{1}{2}-\frac{z^2}{4}\right)v(z) = 0$ are

$$(3.16)\quad D_\lambda(z) = 2^{\frac{\lambda}{2}}\sqrt{\pi}e^{-\frac{z^2}{4}}\left[\frac{1}{\Gamma\left(\frac{1-\lambda}{2}\right)}F\left(-\frac{\lambda}{2},\frac{1}{2};\frac{z^2}{2}\right) - \frac{\sqrt{2}z}{\Gamma\left(-\frac{\lambda}{2}\right)}F\left(\frac{1-\lambda}{2},\frac{3}{2};\frac{z^2}{2}\right)\right]$$

(*Weber's function*), where $F$ is the confluent hypergeometric function. The following $D_{\frac{1}{4}}(z)$, $D_{-\frac{1}{4}}(z)$ are the solutions of (3.12) in Theorem 4.

$$D_{\frac{1}{4}}(z) = 2^{\frac{1}{8}}\sqrt{\pi}e^{-\frac{z^2}{4}}\left[\frac{1}{\Gamma\left(\frac{3}{8}\right)}F\left(-\frac{1}{8},\frac{1}{2};\frac{z^2}{2}\right) - \frac{\sqrt{2}z}{\Gamma\left(-\frac{1}{8}\right)}F\left(\frac{3}{8},\frac{3}{2};\frac{z^2}{2}\right)\right]$$

$$D_{-\frac{1}{4}}(z) = 2^{-\frac{1}{8}}\sqrt{\pi}e^{-\frac{z^2}{4}}\left[\frac{1}{\Gamma\left(\frac{5}{8}\right)}F\left(\frac{1}{8},\frac{1}{2};\frac{z^2}{2}\right) - \frac{\sqrt{2}z}{\Gamma\left(\frac{1}{8}\right)}F\left(\frac{5}{8},\frac{3}{2};\frac{z^2}{2}\right)\right]$$

## References


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# Supplement

We need the following lemma to prove next results.

**Lemma 2** (Miller and Mocanu [5]). Let $h(r,s,t)$ : $\mathbb{C}^3 \longrightarrow \mathbb{C}$ such that

(i) $h(r,s,t)$ is continuous in a domain $D \subset \mathbb{C}^3$,

(ii) $(0,0,0) \in D$ and $|h(0,0,0)| < J$ $(J > 0)$,

(iii) $|h(Je^{i\theta}, Ke^{i\theta}, L)| \geqq J$ when $(Je^{i\theta}, Ke^{i\theta}, L) \in D$, $K \geqq J$ and $\mathrm{Re}[Le^{-i\theta}] \geqq 0$.

Let $w(z) = w_1 z + w_2 z^2 + \cdots$ be analytic in $\Delta$. If $(w(z), zw'(z), z^2 w''(z)) \in D$ $(z \in \Delta)$ and

$$|h(w(z), zw'(z), z^2 w''(z))| < J \quad (z \in \Delta),$$

then $|w(z)| < J$ $(z \in \Delta)$.

**Theorem 5** We consider Airy's differential equation (3.17),

$$v''(z) - z v(z) = 0 \tag{3.17}$$

Then the solution $v(z)$ is starlike.

**Proof** We consider the case of $v(0) = 0$ and $v'(0) = 1$.

We put $u(z) = \frac{zv'(z)}{v(z)} - 1 \quad (z \in \Delta)$, then $u(z)$ is analytic in $\Delta$, $u(0) = 0$ and (3.17) becomes

$$(3.18) \qquad u^2(z) + u(z) + zu'(z) = z^3,$$

or equivalently

$$(3.19) \qquad h(u(z), zu'(z)) = z^3,$$

where $h(r, s) = r^2 + r + s$. It is easy to check $h(r,s) \in H_1$, i.e.,

(i) $h(r, s)$ is continuous in $D \subset \mathbb{C} \times \mathbb{C}$,

(ii) $(0, 0) \in D$, $|h(0, 0)| = 0 < 1$.

(iii) $|h(e^{i\theta}, Ke^{i\theta})| \geq 1 \quad (K \geq 1)$.

By using Theorem 2, we have $|u(z)| < 1$ $(z \in \Delta)$. Therefore, this shows that $\left| \frac{zv'(z)}{v(z)} - 1 \right| < 1$, which implies $\mathrm{Re}\left\{ \frac{zv'(z)}{v(z)} \right\} > 0 \quad (z \in \Delta)$, that is, $v(z)$ is starlike in $\Delta$.

Next we prove the following theorem.

Theorem 6. We consider Weber's differential equation (3.12). Let $\left| z\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) \right| < J \quad (0 < J < 1)$, then we have $\mathrm{Re}\left\{ \frac{v(z)}{z} \right\} > 0 \quad (z \in \Delta)$.

Proof. Put $u(z) = \frac{v(z)}{z} - 1$. Then $u(z)$ is analytic in $\Delta$, $u(0)=0$ and

$$\frac{2zu'(z)}{1+u(z)} + \frac{z^2u''(z)}{1+u(z)} = -z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right)$$

or equivalently

$$h(u(z), zu'(z), z^2u''(z)) = -z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right),$$

where $h(r,s,t) = \frac{2s}{1+r} + \frac{t}{1+r}$.

It is easy to check the following conditions, i.e.,

(i) $h(r,s,t)$ is continuous in $D = \mathbb{C}\setminus\{-1\} \times \mathbb{C} \times \mathbb{C}$,

(ii) $(0,0,0) \in D$ and $|h(0,0,0)| = 0 < J$ $(0 < J < 1)$,

(iii) $|h(Je^{i\theta}, Ke^{i\theta}, L)| \geqq J$ when $(Je^{i\theta}, Ke^{i\theta}, L) \in D$, $K \geqq J$ and $\mathrm{Re}[Le^{-i\theta}] \geqq 0$.

From assumption, we have

$$\left|-z^2\left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right)\right| < J \quad (z \in \Delta).$$

By using Lemma 2, we obtain

$$|u(z)| < J \quad (z \in \Delta).$$

Therefore, we have

$$\mathrm{Re}\left\{\frac{v(z)}{z}\right\} > 0 \quad (z \in \Delta).$$

We need the next lemma to show our last result.

Lemma 3 (Yamaguchi [10]) Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in $\Delta$. If

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > 0 \quad (z \in \Delta), \text{ then we have}$$

$$\mathrm{Re}\{f'(z)\} > 0 \quad \text{for } |z| < \sqrt{2} - 1.$$

By using Lemma 3, we have

Corollary 1 $v(z)$ is close-to-convex in $|z| < \sqrt{2} - 1$.

References

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